A NEW CLASS OF GENERALIZED LUCAS SEQUENCE

Mansi S. Shah[1], Devbhadra V. Shah[2]
Department of Mathematics, Sir P. T. Sarvajanik College of Science, Surat, India.[1]
Department of Mathematics, Sir P. T. Sarvajanik College of Science, Surat, India.[2]

***Abstract**: Lucas sequence* $\{L_n\}$ *is defined by the recurrence relation* $L_n = L_{n-1} + L_{n-2}$ *for* $n \geq 2$ *with initial condition* $L_0 = 2, L_1 = 1$*. This sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. One of the generalizations of the Lucas sequence is the class of sequences* $\left\{G_n^{(a,b)}\right\}$ *generated by the recurrence relation*

$$G_n^{(a,b)} = a\, G_{n-1}^{(a,b)} + b\, G_{n-2}^{(a,b)} \text{ ; for all } n \geq 2,$$

with the initial conditions $G_0^{(a,b)} = 2, G_1^{(a,b)} = a$ *and* a, b *are any positive integers.*
Using the technique of generating functions, we obtain the extended Binet formula for $G_n^{(a,b)}$*.In this paper we express* $G_n^{(a,b)}$ *in simple explicit form and use it to derive the recursive formula for* $G_n^{(a,b)}$ *to compute the approximate value of its successor and predecessor. We also establish some amusing identities for this sequence displaying the relation between* $G_n^{(a,b)}$*, classical Fibonacci sequence & classical Lucas sequence.*

Keywords: **Fibonacci sequence, Generalized Lucas sequence, Lucas sequence.**

I. INTRODUCTION

In the theory of numbers, Fibonacci sequence has always fertile the ground for mathematicians. And Lucas sequence, being its twin sequence, also has this nature. Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with initial conditions $F_0 = 0, F_1 = 1$. Whereas Lucas sequence $\{L_n\}$ is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2, L_1 = 1$. Both these sequences arise naturally in many unexpected places and used in equally surprising places like computer algorithms[3,4,7],some areas of algebra[1,6,8],graph theory[9,14], quasi crystals [5,12] and many areas of mathematics. They occur in a variety of other fields such as finance, art, architecture, music, etc.(See [2, 11]for extensive resources on Fibonacci numbers) The Fibonacci sequence is a source of many identities as appears in the work of Vajda [10] and Harris[13].

Recently, generalization of both the sequences has seized the attention of the mathematicians, which depends on two real parameters used in a recurrence relation. (See [10, 12])

Definition: For any two positive integer *a* and *b*, the generalized Lucas sequence is defined by $G_0^{(a,b)} = 2$, $G_1^{(a,b)} = a$ and $G_n^{(a,b)} = aG_{n-1}^{(a,b)} + bG_{n-2}^{(a,b)}$, where $n \geq 2$.

First few terms of this sequence are $2, a, a^2 + 2b, a^3 + 3ab, a^4 + a^2 b + 4ab + 2b^2, \ldots$. Here it is clear that$G_n^{(1,1)} = L_n$.

II. EXTENDED BINET'S FORMULA FOR $\{G_n^{(a,b)}\}$

In this paper, using the techniques of generating functions, we derive the extended Binet formula for this generalized Lucas sequence and develop some interesting results for this sequence.

Theorem 2.1: The extended Binet's formula for $G_n^{(a,b)}$ is given by $G_n^{(a,b)} = \alpha^n + \beta^n$, where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$.

Proof: Let $g(x) = G_0^{(a,b)} + G_1^{(a,b)} x + G_2^{(a,b)} x^2 + \cdots$ be the generating function for the sequence $\{G_n^{(a,b)}\}$. Then we get

$axg(x) = axG_0^{(a,b)} + aG_1^{(a,b)} x^2 + aG_2^{(a,b)} x^3 + \cdots$and

$bx^2 g(x) = bx^2 G_0^{(a,b)} + bG_1^{(a,b)} x^3 + bG_2^{(a,b)} x^4 + \cdots$. These gives

$$(1 - ax - bx^2)g(x) = G_0^{(a,b)} + \left(G_1^{(a,b)} - aG_0^{(a,b)}\right)x + \left(G_2^{(a,b)} - aG_1^{(a,b)} - bG_2^{(a,b)}\right)x^2 + \cdots$$

Using the definition of $G_n^{(a,b)}$, we get $(1 - ax - bx^2)g(x) = 2 - ax$.

Thus$(x) = \sum_{n=0}^{\infty} G_n^{(a,b)} x^n = \frac{2-ax}{1-ax-bx^2}$, the generating function for $\{G_n^{(a,b)}\}$.

Now let α and β be the roots of the equation$1 - ax - bx^2 = 0$. This gives us $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$. By using the partial fraction, we can write

$$g(x) = \frac{2-ax}{1-ax-bx^2} = \left(\frac{2\alpha-a}{\alpha-\beta}\right)\frac{1}{1-\alpha x} + \left(\frac{2\beta-a}{\beta-\alpha}\right)\frac{1}{1-\beta x}.$$

Now expanding g(x) as power series we get

$g(x) = \left(\frac{2\alpha-a}{\alpha-\beta}\right)\sum_{n=0}^{\infty}\alpha^n x^n + \left(\frac{2\beta-a}{\beta-\alpha}\right)\sum_{n=0}^{\infty}\beta^n x^n$

$$= \sum_{n=0}^{\infty}\left[\left(\frac{2\alpha-a}{\alpha-\beta}\right)\alpha^n - \left(\frac{2\beta-a}{\alpha-\beta}\right)\beta^n\right]x^n$$

$\therefore G_n^{(a,b)} = \frac{(2\alpha-a)\alpha^n-(2\beta-a)\beta^n}{\alpha-\beta}$, which on simplification gives $G_n^{(a,b)} = \alpha^n + \beta^n$.

<u>Remark</u>: We have the following simple results connecting $\alpha$and β which follows immediately from its values.

(1) $\alpha\beta = -b$

(2)$\alpha + \beta = a$

(3)$\alpha - \beta = \sqrt{a^2 + 4b}$

Lemma 2.2: If $2b \leq a$, we have $|\beta| < \frac{1}{2}$.

Proof: Since *a* and *b* are positive integers, we have $2a + 4b > 1$.

$\therefore -2a + 1 < 4b \Rightarrow a^2 - 2a + 1 < a^2 + 4b$

$\Rightarrow (a - 1)^2 < a^2 + 4b$

$\Rightarrow \frac{a-\sqrt{a^2+4b}}{2} < \frac{1}{2}$

$\Rightarrow \beta < \frac{1}{2}$.

Again, $2b \leq a \Rightarrow 4b < 2a+1$

$\Rightarrow a^2+4b < a^2+2a+1$

$\Rightarrow \sqrt{a^2+4b} < a+1$

$\Rightarrow \frac{a-\sqrt{a^2+4b}}{2} > -\frac{1}{2}$

$\Rightarrow \beta > -\frac{1}{2}$

Thus $-\frac{1}{2} < \beta < \frac{1}{2} \Rightarrow |\beta| < \frac{1}{2}$.

We now derive a simple form for $G_n^{(a,b)}$ which gives explicit form for $G_n^{(a,b)}$.

Lemma 2.3: $G_n^{(a,b)} = \lfloor \alpha^n \rfloor$; for $2b \leq a$.

Proof:Using theorem 2.1, we have $\left|G_n^{(a,b)} - \alpha^n\right| = |\beta^n| = |\beta|^n$.

But when $2b \leq a$, by lemma 2.2 we have$|\beta| < \frac{1}{2}$.Thus $\left|G_n^{(a,b)} - \alpha^n\right| < \frac{1}{2}$.

Hence $G_n^{(a,b)} = \lfloor \alpha^n \rfloor$.

III. SOME FUNDAMENTAL IDENTITIES FOR $\{G_n^{(a,b)}\}$

We obtain a nice result for the product of two consecutive generalized Lucas numbers.

Lemma3.1: $G_n^{(a,b)} G_{n+1}^{(a,b)} = G_{2n+1}^{(a,b)} + (-1)^n ab^n$.

Proof: $G_n^{(a,b)} G_{n+1}^{(a,b)} = \{\alpha^n + \beta^n\}\{\alpha^{n+1} + \beta^{n+1}\}$

$= \alpha^n\alpha^{n+1} + \alpha^n\beta^{n+1} + \alpha^{n+1}\beta^n + \beta^n\beta^{n+1}$

$= \alpha^{2n+1} + \beta^{2n+1} + \alpha^n\beta^n(\alpha+\beta)$

Since $\alpha\beta = -b$ and $\alpha+\beta = a$, we have $G_n^{(a,b)} G_{n+1}^{(a,b)} = G_{2n+1}^{(a,b)} + (-1)^n ab^n$.

The following identity for the square of generalized Lucas number is an easy consequence of its extended Binet's formula.

Lemma3.3: $\left(G_n^{(a,b)}\right)^2 = G_{2n}^{(a,b)} + (-1)^n 2b^n$.

Proof: $\left(G_n^{(a,b)}\right)^2 = (\alpha^n + \beta^n)^2 = \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n}$.

Since $\alpha\beta = -b$, we have $\left(G_n^{(a,b)}\right)^2 = G_{2n}^{(a,b)} + (-1)^n 2b^n$.

We next derive the result for the value of $G_{n+1}^{(a,b)}$ when the value of $G_n^{(a,b)}$ is known.

Lemma 3.4: $G_{n+1}^{(a,b)} = \alpha G_n^{(a,b)} + \delta$; $|\delta| < \sqrt{a^2+4b}$,if$2b \leq a$.

Proof: We have $G_n^{(a,b)} = \alpha^n + \beta^n$. This gives

$G_{n+1}^{(a,b)} = \alpha^{n+1} + \beta^{n+1}$

$= \alpha\left(G_n^{(a,b)} - \beta^n\right) + \beta^{n+1}$

$= \alpha G_n^{(a,b)} + \beta^n(\beta - \alpha)$

$\therefore G_{n+1}^{(a,b)} - \alpha G_n^{(a,b)} = \beta^n(\beta - \alpha)$.

Now when $2b \leq a$, we know that $|\beta| < \frac{1}{2} < 1$.

$\therefore \left|G_{n+1}^{(a,b)} - \alpha G_n^{(a,b)}\right| < |\alpha - \beta| < \left|\sqrt{a^2 + 4b}\right|$

$\Rightarrow G_{n+1}^{(a,b)} = \alpha G_n^{(a,b)} + \delta$; where= $\sqrt{a^2 + 4b}$.

Using the same result, we can derive the formula to find $G_n^{(a,b)}$ when the value of $G_{n+1}^{(a,b)}$ is known.

Corollary 3.5: $G_n^{(a,b)} = \frac{1}{\alpha}[G_{n+1}^{(a,b)} - \delta]$; when $2b \leq a$ and $\delta = \sqrt{a^2 + 4b}$.

Proof: The result follows from the above lemma.

Corollary 3.6: $\lim_{n\to\infty} G_{n+1}^{(a,b)} = \alpha G_n^{(a,b)}$; when $2b \leq a$.

Proof: Since $|\beta| < \frac{1}{2}$, the result follows immediately from above lemma.

IV. VALUE OF α^n

We first define the generalized Fibonacci numbers.

Definition: For any two positive integer *a* and *b*, the generalized Fibonacci sequence $\left\{F_n^{(a,b)}\right\}$ is defined by $F_0^{(a,b)} = 0$, $F_1^{(a,b)} = 1$ and $F_n^{(a,b)} = aF_{n-1}^{(a,b)} + bF_{n-2}^{(a,b)}$, where $n \geq 2$.

It can be shown that by method of theorem 2.1 that the related extended Binet formula is $F_n^{(a,b)} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

We first find an expression for the value of α^n in terms of $F_n^{(a,b)}$ and $G_n^{(a,b)}$.

Theorem 4.1: $\alpha^n = \frac{G_n^{(a,b)} + \sqrt{a^2+4b}F_n^{(a,b)}}{2}$.

Proof: We use the principal of mathematical induction to prove this result.For $n = 1$, the right side of expression becomes $\frac{G_1^{(a,b)} + \sqrt{a^2+4b}F_1^{(a,b)}}{2} = \frac{a+\sqrt{a^2+4b}}{2} = \alpha$, which proves the result for $n = 1$. Assume that the result is true for all positive integers not exceeding the positive integer *m*.i.e. we assume that $\alpha^m = \frac{G_m^{(a,b)} + \sqrt{a^2+4b}F_m^{(a,b)}}{2}$ holds.

Now, $\frac{G_{m+1}^{(a,b)} + \sqrt{a^2+4b}F_{m+1}^{(a,b)}}{2} = \frac{1}{2}\left[aG_m^{(a,b)} + bG_{m-1}^{(a,b)} + \sqrt{a^2+4b}\left(aF_m^{(a,b)} + bF_{m-1}^{(a,b)}\right)\right]$

$$= a\left(\frac{G_m^{(a,b)} + \sqrt{a^2+4b}F_m^{(a,b)}}{2}\right) + b\left(\frac{G_{m-1}^{(a,b)} + \sqrt{a^2+4b}F_{m-1}^{(a,b)}}{2}\right)$$

$= a\ \alpha^m + b\ \alpha^{m-1}$

$= \alpha^{m-1}\ [a\alpha + b]$

$= \alpha^{m-1}\alpha^2$

$= \alpha^{m+1}$, which proves the result for $n = m + 1$ also.

Hence by the principle of mathematical induction, the result is true for every positive integers *n*.

We next find two results which expresses α^n in terms of two consecutive value of $G_n^{(a,b)}$.

Theorem 4.2: $\alpha^n = \gamma\left[G_n^{(a,b)}\alpha + bG_{n-1}^{(a,b)}\right]$, where $\gamma = \frac{\alpha}{a\alpha+2b}$.

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Proof: Here tooweuse the principal of mathematical induction to prove this result. For $n = 1$, the right side gives, $\gamma\left[G_1^{(a,b)}\alpha + bG_0^{(a,b)}\right] = \frac{\alpha}{a\alpha+2b}[a\alpha + 2b] = \alpha$, which proves the result for $n = 1$.

Now suppose the result holds for some positive integer$n = m$.

i.e. we assume that$\alpha^m = \gamma\left(G_m^{(a,b)}\alpha + bG_{m-1}^{(a,b)}\right)$ holds.

Now, $\alpha^{m+1} = \alpha\alpha^m$

$= \alpha\gamma\left(G_m^{(a,b)}\alpha + bG_{m-1}^{(a,b)}\right)$

$= \gamma\left(\alpha^2 G_m^{(a,b)} + \alpha bG_{m-1}^{(a,b)}\right)$

$= \gamma\left((a\alpha + b)G_m^{(a,b)} + \alpha bG_{m-1}^{(a,b)}\right)$

$= \gamma\left(\alpha\left(aG_m^{(a,b)} + bG_{m-1}^{(a,b)}\right) + bG_m^{(a,b)}\right)$

$= \gamma\left(\alpha G_{m+1}^{(a,b)} + bG_m^{(a,b)}\right)$, which proves the result for $n = m + 1$ also.

Hence by the principle of mathematical induction the result is true for every positive integer n.

Lemma 4.3: $\alpha^n = \frac{G_{n+1}^{(a,b)} - \beta G_n^{(a,b)}}{\mu}$, where $\mu = \sqrt{a^2 + 4b}$.

Proof:$G_n^{(a,b)} = \alpha^n + \beta^n$

$\Rightarrow \alpha G_n^{(a,b)} = \alpha^{n+1} + \alpha\beta^n$

$\Rightarrow \alpha G_n^{(a,b)} - \mu\beta^n = \alpha^{n+1} + \beta^n(\alpha - \mu)$

$= \alpha^{n+1} + \beta^{n+1}$

$= G_{n+1}^{(a,b)}$.

Also using the fact that $G_n^{(a,b)} = \alpha^n + \beta^n$, we get $\alpha^n = \frac{G_{n+1}^{(a,b)} - \beta G_n^{(a,b)}}{\mu}$.

Lemma 4.4:$\beta^n = \frac{G_{n+1}^{(a,b)} - \alpha G_n^{(a,b)}}{\mu}$, where $\mu = \sqrt{a^2 + 4b}$.

This result can be proved easily by the method used in lemma 4.3.

V. SOME MORE IDENTITIES

Here we develop the results which relates $G_n^{(a,b)}$ with the generalized Fibonacci sequence $F_n^{(a,b)}$.

Lemma 5.1:$G_n^{(a,b)} = G_{n-k}^{(a,b)}F_{k+1}^{(a,b)} + bG_{n-k-1}^{(a,b)}F_k^{(a,b)}$.

Proof: We have $G_n^{(a,b)} = \alpha^n + \beta^n$.

$\therefore G_{n-2}^{(a,b)}G_n^{(a,b)} - \left(G_{n-1}^{(a,b)}\right)^2 = (\alpha^{n-2} + \beta^{n-2})(\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})^2$

$= \alpha^{2n-2} + \alpha^{n-2}\beta^{n-2}(\alpha^2 + \beta^2) + \beta^{2n-2} - \alpha^{2n-2} - 2(\alpha\beta)^{n-1} - \beta^{2n-2}$

$= (\alpha\beta)^{n-2}(a^2 + 2b) - 2(\alpha\beta)^{n-1}$

$= (\alpha\beta)^{n-2}(a^2 + 2b - 2\alpha\beta)$

$$= (-b)^{n-2}(a^2 + 4b).$$

Again, $G_n^{(a,b)} = \alpha^n + \beta^n$

$$\Rightarrow \begin{pmatrix} G_{n-k}^{(a,b)} \\ G_{n-k-1}^{(a,b)} \end{pmatrix} = \begin{pmatrix} \alpha^{n-k} & \beta^{n-k} \\ \alpha^{n-k-1} & \beta^{n-k-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \text{ where } c_1 = c_2 = 1.$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha^{n-k} & \beta^{n-k} \\ \alpha^{n-k-1} & \beta^{n-k-1} \end{pmatrix}^{-1} \begin{pmatrix} G_{n-k}^{(a,b)} \\ G_{n-k-1}^{(a,b)} \end{pmatrix}$$

$$= \frac{1}{\alpha^{n-k}\beta^{n-k-1} - \alpha^{n-k-1}\beta^{n-k}} \begin{pmatrix} \beta^{n-k-1} & -\beta^{n-k} \\ -\alpha^{n-k-1} & \alpha^{n-k} \end{pmatrix} \begin{pmatrix} G_{n-k}^{(a,b)} \\ G_{n-k-1}^{(a,b)} \end{pmatrix}$$

$$= \frac{1}{(\alpha\beta)^{n-k-1}(\alpha - \beta)} \begin{pmatrix} G_{n-k}^{(a,b)}\beta^{n-k-1} - \beta^{n-k}G_{n-k-1}^{(a,b)} \\ -\alpha^{n-k-1}G_{n-k}^{(a,b)} + \alpha^{n-k}G_{n-k-1}^{(a,b)} \end{pmatrix}$$

This gives

$$c_1 = \frac{G_{n-k}^{(a,b)}\beta^{n-k-1} - \beta^{n-k}G_{n-k-1}^{(a,b)}}{(\alpha\beta)^{n-k-1}(\alpha-\beta)} \text{ and } c_2 = \frac{-\alpha^{n-k-1}G_{n-k}^{(a,b)} + \alpha^{n-k}G_{n-k-1}^{(a,b)}}{(\alpha\beta)^{n-k-1}(\alpha-\beta)}$$

Since $c_1 = c_2 = 1$, we have $\frac{G_{n-k}^{(a,b)} - \beta.G_{n-k-1}^{(a,b)}}{\alpha^{n-k-1}(\alpha-\beta)} = 1$ and $\frac{-G_{n-k}^{(a,b)} + \alpha.G_{n-k-1}^{(a,b)}}{\beta^{n-k-1}(\alpha-\beta)} = 1$.

Thus we write $G_n^{(a,b)} = c_1\alpha^n + c_2\beta^n$

$$= \left(\frac{G_{n-k}^{(a,b)} - \beta G_{n-k-1}^{(a,b)}}{\alpha^{n-k-1}(\alpha - \beta)}\right)\alpha^n + \left(\frac{-G_{n-k}^{(a,b)} + \alpha G_{n-k-1}^{(a,b)}}{\beta^{n-k-1}(\alpha - \beta)}\right)\beta^n$$

$$= \frac{\alpha^{k+1}\left(G_{n-k}^{(a,b)} - \beta.G_{n-k-1}^{(a,b)}\right) + \beta^{k+1}\left(-G_{n-k}^{(a,b)} + \alpha.G_{n-k-1}^{(a,b)}\right)}{\alpha - \beta}$$

$$= G_{n-k}^{(a,b)}\left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}\right) - G_{n-k-1}^{(a,b)}(\alpha\beta)\left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\right)$$

This gives $G_n^{(a,b)} = G_{n-k}^{(a,b)}F_{k+1} + bG_{n-k-1}^{(a,b)}F_k$, since $\alpha\beta = -b$.

We next express $G_n^{(a,b)}$ in terms of binomial coefficients using the Pascal's identity.

Lemma 5.2: $G_n^{(a,b)} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A(n,i)a^{n-2i}b^i$, where $A(n,i) = \frac{n}{n-i}\binom{n-i}{i}$.

Proof: We know that $\alpha + \beta = a$ and $\alpha\beta = -b$.

Now we observe that $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = a^2 + 2b$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = a^3 + 2ab$

$\alpha^4 + \beta^4 = a^4 + 4a^2b + 2b^2$

Continuing in this way, we can write

$\alpha^n + \beta^n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A(n,i)a^{n-2i}b^i$, where $A(n,i) = 2\binom{n-i}{i} - \binom{n-i-1}{i}$.

Now using Pascal's identity $A(n,i) = \frac{n}{n-i}\binom{n-i}{i}$, we get

$\alpha^n + \beta^n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A(n,i)a^{n-2i}b^i$.

Thus $G_n^{(a,b)} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A(n,i)a^{n-2i}b^i$.

Lemma 5.3: $G_n^{(a,b)} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{1}{2} \rfloor} \binom{n}{2i} a^{n-2i} \mu^{2i}$, where $\mu = \sqrt{a^2+4b}$.

Proof: If $\mu = \sqrt{a^2+4b}$then $2\alpha = a + \mu$ and $2\beta = a - \mu$.

$$\therefore (2\alpha)^n + (2\beta)^n = 2 \sum_{i \text{ even}} \binom{n}{i} a^{n-i} \mu^i$$

$$\therefore 2^n(\alpha^n + \beta^n) = 2 \sum_{i=0}^{\lfloor \frac{1}{2} \rfloor} \binom{n}{2i} a^{n-2i} \mu^{2i}$$

Hence, $G_n^{(a,b)} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{1}{2} \rfloor} \binom{n}{2i} a^{n-2i} \mu^{2i}$.

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